Echo–Silence on the Critical Horizon and the Riemann Hypothesis

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Scope and Claims

Unconditional. We define the mirror functional $\mathcal{E}_{\sigma,\Lambda,y}(T)$ and prove its exact residue expansion converges absolutely via Gaussian decay (Lemma 2.3).
Conditional (RH). We prove the equivalence: RH holds if and only if $\mathcal{E}_{\sigma,\Lambda,y}(T) = o(T^{1/2-\sigma})$ uniformly for y in any compact subinterval of a bounded interval.
Not claimed. This paper does not prove the required vanishing bound unconditionally. The companion paper establishes this via coprime-diagonal methods.

Parameter audit. Fix once: $\sigma \in (\frac{1}{2}, 1)$, Gaussian scale $\Lambda > 0$, tilt parameter $y \in \mathbb{R}$, scale parameter $T \geq 2$. The mirror weight $W_{\Lambda,y}(s) = \exp((s-\frac{1}{2})^2/\Lambda^2) \cdot \exp(y(s-\frac{1}{2}))$ ensures $|W_{\Lambda,y}(\sigma + it)| \ll e^{-t^2/\Lambda^2}$ on verticals. At zeros $\rho = \beta + i\gamma$, $|W_{\Lambda,y}(\rho)| \ll e^{-\gamma^2/\Lambda^2}$ (bounded uniformly in $\beta \in [0,1]$). All sums over zeros use symmetric height truncation $|\gamma| \leq U$ with $U \to \infty$.

Parameter Table

T	height parameter (tends to ∞)
σ	fixed in $(\frac{1}{2}, 1)$ (later $\sigma \in [\frac{1}{2}+\kappa, 1-\kappa]$)
Λ	Gaussian scale in the s-plane (fixed, large enough)
y	tilt parameter for the mirror test (y in a fixed bounded interval)
$W_{\Lambda,y}(s)$	even-in-$s - \frac{1}{2}$ Gaussian with tilt (Remark 1.2)

Zeros and residues. Since all trivial zeros of ζ lie at negative even integers, they are strictly to the left of $\Re s = 0$. For $\sigma \in (\frac{1}{2}, 1)$ the vertical lines $\Re s = \sigma$ and $\Re s = 1 - \sigma$ therefore enclose only the nontrivial zeros. Moreover,

$$\left| W_{\Lambda,y}(\sigma + it) \right| = \exp\!\Big(\tfrac{(\sigma-\frac{1}{2})^2 - t^2}{\Lambda^2}\Big)\, e^{y(\sigma-\frac{1}{2})},$$

so the Gaussian factor yields absolute convergence of the residue expansion and justifies letting any symmetric height truncation $|\gamma| \leq U$ tend to infinity uniformly for y in any compact subinterval of a bounded interval.

Abstract

We define a *mirror functional* $\mathcal{E}_{\sigma,\Lambda,y}(T)$ with tilt and scale parameters (y, Λ) for the completed zeta $\xi(s)$ which, for each $\sigma \in (\frac{1}{2}, 1)$, Gaussian scale $\Lambda > 0$, and tilt $y \in \mathbb{R}$, expands as

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \sum_{\rho} m(\rho) \left[W_{\Lambda,y}(\rho)\, T^{\beta-\frac{1}{2}} - W_{\Lambda,-y}(\rho)\, T^{\frac{1}{2}-\beta} \right] \;+\; O_{\sigma,\Lambda}(T^{-A})$$

where the sum runs over nontrivial zeros $\rho = \beta + i\gamma$ of ζ, $m(\rho)$ is the multiplicity, and $W_{\Lambda,y}$ is an entire, nonvanishing weight satisfying $W_{\Lambda,y}(1-s) = W_{\Lambda,-y}(s)$. We prove:

(i) If the Riemann Hypothesis holds, then $\mathcal{E}_{\sigma,\Lambda,y}(T) \equiv 0$ for all $\sigma \in (\frac{1}{2}, 1)$, $\Lambda > 0$, $y \in \mathbb{R}$, and $T \geq 2$.

(ii) Conversely, if for some fixed $\sigma \in (\frac{1}{2}, 1)$ and $\Lambda > 0$ there exists a nonempty open interval $I \subset \mathbb{R}$ such that

$$\forall\, y \in I: \qquad \mathcal{E}_{\sigma,\Lambda,y}(T) = o\big(T^{\frac{1}{2}-\sigma}\big) \qquad (T \to \infty),$$

then all nontrivial zeros of ζ lie on the critical line $\Re s = \frac{1}{2}$.

Thus *uniform echo–silence across mirrors* (in one real tilt parameter) is equivalent to RH. The argument uses only classical complex analysis (functional equation for ξ, Stirling on vertical lines, contour shifts, residues).

1 Introduction

Let

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma\big(\tfrac{s}{2}\big)\,\zeta(s),$$

so ξ is entire of order 1, satisfies the functional equation $\xi(s) = \xi(1-s)$, and its zeros coincide with the nontrivial zeros of ζ (the factor $s(s-1)$ absorbs 0 and 1). Write $J(s) := 1 - s$.

The principle is simple: any zero $\rho = \beta + i\gamma$ off the critical line produces a detectable imbalance between the mirror lines $\Re s = \sigma$ and $\Re s = 1 - \sigma$ in the monomials $T^{\beta-\sigma}$ versus $T^{1-\beta-\sigma}$. A Gaussian weight ensures uniform decay on verticals, while a real *tilt* y eliminates accidental cancellation of finitely many equal-exponent terms through a short exponential-sum argument.

Main result

Theorem 1.1 (Echo–Silence $\iff$ RH)**.** *Fix $\Lambda > 0$ and define*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \frac{1}{2\pi i}\left(\int_{\Re s = \sigma} - \int_{\Re s = 1-\sigma}\right)\frac{\xi'}{\xi}(s)\,T^{\,s-\frac{1}{2}}\,W_{\Lambda,y}(s)\,ds, \qquad W_{\Lambda,y}(s) := \exp\Big(\frac{(s-\frac{1}{2})^2}{\Lambda^2}\Big)e^{\,y(s-\frac{1}{2})}.$$

Convention. *If a zero of $\xi(s)$ lies on a boundary line, the contour is indented by a tiny semicircle to avoid the singularity. The contribution of such indentations vanishes by the vertical Gaussian bounds (see Lemma 2.1).*

Remark 1.2 (Weight symmetry)**.** Since $(1 - s - \frac{1}{2})^2 = (s - \frac{1}{2})^2$, we have

$$W_{\Lambda,y}(1-s) \;=\; e^{-y(s-\frac{1}{2})}\,e^{(s-\frac{1}{2})^2/\Lambda^2}, \qquad \text{hence} \qquad W_{\Lambda,-y}(1-s) = W_{\Lambda,y}(s).$$

We will use this identity when comparing the mirror integrals on $\Re s = \sigma$ and $\Re s = 1 - \sigma$.

The following are equivalent:

(i) *The Riemann Hypothesis holds.*

(ii) *For every $\sigma \in (\frac{1}{2}, 1)$ there exists a nonempty open interval $I_\sigma \subset \mathbb{R}$ such that*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = o\big(T^{\frac{1}{2}-\sigma}\big) \qquad \text{as } T \to \infty,$$

uniformly for $y \in I_\sigma$.

Sketch. Under RH, the open strip $1-\sigma < \Re s < \sigma$ contains no zeros of ξ, so the contour difference encloses no poles and $\mathcal{E}_{\sigma,\Lambda,y}(T) \equiv 0$.

Conversely, suppose (ii) holds but RH fails. Let $\beta^* = \sup\{\Re\rho\} > 1/2$. Choose σ with $1-\sigma < \beta^* < \sigma$. The residue expansion gives

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = T^{\beta^*-\frac{1}{2}} F(y) + o\big(T^{\beta^*-\frac{1}{2}}\big),$$

where $F(y)$ is a nontrivial exponential polynomial in y built from zeros with real part β^*. Since $\beta^* - \frac{1}{2} > \frac{1}{2} - \sigma$, the uniform $o(T^{\frac{1}{2}-\sigma})$ contradicts this unless $F \equiv 0$, impossible. Hence RH holds.

Remark 1.3 (Why the parameters)**.** The Gaussian delivers rapid decay on verticals, uniform in T; the tilt $e^{y(s-1/2)}$ is entire and yields Fourier weights $e^{iy\gamma}$ on ordinates γ, letting us rule out accidental linear cancellation on the leading front by varying y over an open interval. Note $W_{\Lambda,y}(1-s) = W_{\Lambda,-y}(s)$.

2 Preliminaries and vertical bounds

We use standard facts (see Titchmarsh–Heath-Brown or Iwaniec–Kowalski).

Lemma 2.1 (Bounds on vertical lines)**.** *Fix $\sigma \in (0,1)$. For $|t| \geq 2$,*

$$\frac{\xi'}{\xi}(\sigma + it) \ll \log(|t|+2).$$

The weight bound follows immediately from the Gaussian decay of $W_{\Lambda,y}$.

Proof. From $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, Stirling gives $\frac{d}{ds}\log\Gamma(s/2) \ll \log(|t|+2)$, while ζ'/ζ is $\ll \log(|t|+2)$ away from $s=1$; combine. The weight bound follows from the Gaussian factor in $W_{\Lambda,y}$.

Lemma 2.2 (Poles and residues)**.** *If ρ is a zero of ζ of multiplicity $m(\rho)$, then $\mathrm{Res}_{s=\rho}\frac{\xi'}{\xi}(s) = m(\rho)$. The same holds at $1-\rho$ by the functional equation.*

Lemma 2.3 (Absolute convergence of the residue expansion)**.** *For fixed $\Lambda > 0$ and $|y| \leq Y$,*

$$\sum_\rho |W_{\Lambda,y}(\rho)| \;<\; \infty,$$

hence (3) *converges absolutely and locally uniformly in* (T,y)*.*

Proof. At $\rho = \beta + i\gamma$, we have $|W_{\Lambda,y}(\rho)| = \exp((\beta-\frac{1}{2})^2/\Lambda^2 - \gamma^2/\Lambda^2)\cdot e^{y(\beta-\frac{1}{2})} \leq e^{1/(4\Lambda^2)}e^{Y/2}e^{-\gamma^2/\Lambda^2}$. Group zeros by $k < |\gamma| \leq k+1$; with $N(k+1) - N(k) \ll \log k$, we get

$$\sum_\rho |W_{\Lambda,y}(\rho)| \ll \sum_{k=1}^\infty (\log k)\, e^{-k^2/\Lambda^2} < \infty.$$

Convergence. The Gaussian vertical decay of $W_{\Lambda,y}$ yields $\sum_\rho |W_{\Lambda,y}(\rho)| < \infty$ uniformly for y in compact sets, so all rearrangements are justified without appealing to zero-density estimates.

Summation convention over zeros. Every sum $\sum_\rho$ over nontrivial zeros is defined as the limit of **symmetric height truncations**:

$$\sum_\rho A(\rho) \;:=\; \lim_{U\to\infty} \sum_{\substack{\rho=\beta+i\gamma \\ |\gamma|\le U}} A(\rho),$$

whenever the limit exists. In our setting, the limit exists because the contour integrals in (1) converge absolutely on the vertical lines (Lemma 2.1), the horizontal edges vanish as $U\to\infty$, and the **residue theorem equates** the truncated sum to the difference of the two vertical integrals up to an error $o(1)$ which tends to 0 as $U\to\infty$. All **uniformity in** y (on compact intervals) is derived from the **integral bounds**.

Boundary zeros. If a zero lies on $\Re s=\sigma$ or $\Re s = 1-\sigma$, we use an indentation or principal value; the residue contribution is then half-weighted in the usual way. This has no effect on the asymptotics and will be omitted from notation.

3 Contour architecture and exact residue expansion

Let $U\ge 2$. Consider the rectangle $\mathcal{R}_U$ with vertical edges $\Re s=\sigma$ and $\Re s=1-\sigma$ and horizontal edges $\Im s=\pm U$. Define

$$F(s) := \frac{\xi'}{\xi}(s)\,T^{\,s-\frac12}\,W_{\Lambda,y}(s).$$

By Lemma 2.1, $|F(\sigma+it)| \ll e^{-(t/\Lambda)^2}\log(|t|+2)$ and similarly on $\Re s = 1-\sigma$. The contributions of the horizontal edges are

$$\ll \int_{1-\sigma}^{\sigma} e^{-(U/\Lambda)^2}\log(U+2)\,dx \;\ll\; e^{-(U/\Lambda)^2}\log(U+2),$$

hence vanish as $U\to\infty$. Thus by Cauchy's theorem,

$$\begin{aligned}\mathcal{E}_{\sigma,\Lambda,y}(T) &= \frac{1}{2\pi i}\int_{\partial\mathcal{R}_U} F(s)\,ds \\ &= \sum_{\rho\in\mathcal{Z}_U} \mathrm{Res}_{s=\rho}\,F(s),\end{aligned} \tag{1}$$

where $\mathcal{Z}_U$ are the zeros of ξ in the strip $1-\sigma<\Re s<\sigma$, $|\Im s|<U$.

Each residue equals $m(\rho)\,T^{\rho-\frac12}W_{\Lambda,y}(\rho)$ by Lemma 2.2. Passing $U\to\infty$ (dominated convergence holds by Lemma 2.1), we obtain an *exact* expansion:

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \sum_\rho m(\rho)\,T^{\,\rho-\frac12}\,W_{\Lambda,y}(\rho), \tag{2}$$

where the sum runs over *all* nontrivial zeros ρ with $1-\sigma<\Re\rho<\sigma$.

Proposition 3.1 (Global mirror expansion)**.** *For $\sigma\in(\frac12,1)$, $\Lambda>0$, $T\ge 2$ and any $y\in\mathbb{R}$,*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \sum_\rho m(\rho)\Big(T^{\rho-\frac12}\,W_{\Lambda,y}(\rho) \;-\; T^{\frac12-\rho}\,W_{\Lambda,-y}(\rho)\Big), \tag{3}$$

where the sum is over all nontrivial zeros ρ of ζ. The series converges absolutely and locally uniformly in (T,y).

Proof sketch. Starting from (1), change variables $s \mapsto 1-s$ in the second integral and use $\xi(1-s) = \xi(s)$ and $\frac{\xi'}{\xi}(1-s) = -\frac{\xi'}{\xi}(s)$ to obtain

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \frac{1}{2\pi i}\int_{\Re s=\sigma} \frac{\xi'}{\xi}(s)\Big(T^{s-\frac{1}{2}}W_{\Lambda,y}(s) - T^{\frac{1}{2}-s}W_{\Lambda,y}(1-s)\Big)\,ds.$$

By the weight symmetry $W_{\Lambda,y}(1-s) = W_{\Lambda,-y}(s)$ this is

$$\frac{1}{2\pi i}\int_{\Re s=\sigma} \frac{\xi'}{\xi}(s)\Big(T^{s-\frac{1}{2}}W_{\Lambda,y}(s) - T^{\frac{1}{2}-s}W_{\Lambda,-y}(s)\Big)\,ds.$$

Both terms have Gaussian decay on vertical lines (Lemma 2.1), so we may shift the line and apply the residue theorem. Each zero ρ contributes

$$\mathrm{Res}_{s=\rho}\,\frac{\xi'}{\xi}(s)\Big(T^{s-\frac{1}{2}}W_{\Lambda,y}(s) - T^{\frac{1}{2}-s}W_{\Lambda,-y}(s)\Big) = m(\rho)\Big(T^{\rho-\frac{1}{2}}W_{\Lambda,y}(\rho) - T^{\frac{1}{2}-\rho}W_{\Lambda,-y}(\rho)\Big),$$

and grouping by real parts yields (3). Absolute convergence follows from Lemma 2.3 applied to both weights and the standard zero count $N(T) \asymp T\log T$.

Corollary 3.2 (Phaseful $y=0$ form)**.** *By pairing ρ with $1-\rho$ in* (3)*, one recovers the strip form*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \sum_{1-\sigma<\beta<\sigma} m(\rho)\,T^{\rho-\frac{1}{2}}\,W_{\Lambda,y}(\rho),$$

and when $y=0$,

$$\mathcal{E}_{\sigma,\Lambda,0}(T) = T^{\sigma-\frac{1}{2}}\sum_{\rho} m(\rho)\,W_{\Lambda,0}(\rho)\Big(T^{\beta-\sigma}e^{i\gamma\log T} - T^{1-\beta-\sigma}e^{-i\gamma\log T}\Big).$$

Pairing conjugate zeros yields a real trigonometric form.

Remark 3.3 (Why the $-y$ appears)**.** The second term in (3) necessarily carries $W_{\Lambda,-y}$ because the change of variables $s \mapsto 1-s$ flips the tilt: $W_{\Lambda,y}(1-s) = W_{\Lambda,-y}(s)$. In the *converse* direction (Section 5), only the first term contributes to the dominant exponent $T^{\beta^{-\frac{1}{2}}}$ when $\beta^{=\sup\Re\rho>1/2}$, so the $-y$ never affects the leading coefficient $F_{\Lambda,U}(y)$ used in the exponential-polynomial argument.

Remark 3.4 (Remainder term)**.** If one prefers to shift just the $\Re s = \sigma$ line to $+\infty$ and $\Re s = 1-\sigma$ to $-\infty$, one obtains (3) with an exponentially small $O_{\sigma,\Lambda}(T^{-A})$ remainder coming from the tails; we may harmlessly record this as an $O(T^{-A})$ term in all that follows.

4 RH ⇒ echo–silence

Proposition 4.1 (RH ⇒ echo–silence)**.** *Fix $\sigma \in (1/2,1)$ and $y \in \mathbb{R}$. Assuming RH, the open strip $1-\sigma < \Re s < \sigma$ contains no zeros of ζ, hence*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) \equiv 0 \qquad \text{for all } T \geq 2.$$

Proof. Under RH there are no poles of ξ'/ξ in $1-\sigma < \Re s < \sigma$. The difference of the two vertical integrals is a closed contour integral enclosing no poles; horizontal segments vanish by the superpolynomial vertical decay of $W_{\Lambda,y}$, so Cauchy's theorem gives 0.

5 Echo–silence on a tilt interval ⇒ RH

Theorem 5.1 (Asymptotics from an off-line zero)**.** *Fix $\sigma \in (1/2, 1)$ and $U \geq 1$. Suppose RH fails and let*

$$\beta^* = \sup\{\Re\rho : \ \zeta(\rho) = 0\} > 1/2.$$

Then for any σ with $1 - \sigma < \beta^* < \sigma$,

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = T^{\beta^* - \frac{1}{2}} \, F_{\Lambda,U}(y) \ + \ o\big(T^{\beta^* - \frac{1}{2}}\big) \quad (T \to \infty),$$

uniformly for y in compact sets, where

$$F_{\Lambda,U}(y) := \sum_{\substack{\rho:\ \Re\rho = \beta^* \\ |\Im\rho| \leq U}} m(\rho) \, W_{\Lambda,y}(\rho) = e^{y(\beta^* - \frac{1}{2})} \sum_{\substack{\rho:\ \Re\rho = \beta^* \\ |\Im\rho| \leq U}} m(\rho) \exp\Big(\frac{(\rho - \frac{1}{2})^2}{\Lambda^2}\Big) \, e^{iy\,\Im\rho}$$

is a nontrivial exponential polynomial in y (after fixing U).

Proof sketch. From the phaseful global mirror expansion

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \sum_{\rho} m(\rho)\Big(T^{\rho - \frac{1}{2}} W_{\Lambda,y}(\rho) - T^{\frac{1}{2} - \rho} W_{\Lambda,-y}(\rho)\Big),$$

the terms with $\Re\rho = \beta^*$ dominate, contributing $T^{\beta^* - \frac{1}{2}}$ up to phases $e^{\pm i(\Im\rho)\log T}$. Truncating $|\Im\rho| \leq U$ produces $F_{\Lambda,U}(y)$; the tail is $o(1)$ by the Gaussian vertical decay and standard zero counting. Nontriviality holds because $W_{\Lambda,\cdot}$ is entire, nonvanishing and the residue of ξ'/ξ at ρ is 1.

Lemma 5.2 (Exponential polynomial in y)**.** *For fixed $U \geq 1$, the function $F_{\Lambda,U}(y)$ in Theorem 5.1 is a finite sum $\sum_{j=1}^{J} c_j e^{i\gamma_j y}$ with distinct real γ_j and nonzero coefficients c_j depending on $W_{\Lambda,\cdot}(\rho_j)$.*

Remark 5.3 (Multiplicity and coincident ordinates)**.** If several zeros share the same ordinate γ, their coefficients aggregate in the corresponding exponential term. All arguments below apply to the grouped coefficient; in particular, the nondegeneracy of the exponential polynomial on any open y-interval does not require a simplicity assumption on zeros.

Assume RH fails; let

$$\beta^{:=\sup\{\Re\rho:\ \zeta(\rho)=0\}>\frac{1}{2}}.$$

Let $\mathcal{Z}_U^{:=\{\rho:\Re\rho=\beta,|\Im\rho|\leq U\}}$ (finite for each U). From the global mirror expansion (3), when $\beta^{>\frac{1}{2}}$, the dominant contribution comes from zeros with $\Re\rho = \beta$ in the first term (the second term with $W_{\Lambda,-y}$ contributes only $O(T^{-(\beta+\sigma-1)})$). Thus

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = \lim_{U\to\infty}\Big[T^{\beta^{-\frac{1}{2}}\sum_{\rho\in\mathcal{Z}_U} m(\rho)\,W_{\Lambda,y}(\rho) \ + \ \sum_{\substack{\beta<\beta \\ |\gamma|\leq U}} m(\rho)\,W_{\Lambda,y}(\rho)\,T^{\beta-\frac{1}{2}} \ - \ \sum_{|\gamma|\leq U} m(\rho)\,W_{\Lambda,-y}(\rho)\,T^{\frac{1}{2}-\beta}}\Big]. \quad (4)$$

the last sum contributes $o(T^{\beta^{-\frac{1}{2}}})$ since $\frac{1}{2} - \beta^{<0}$. Likewise the middle sum is $o(T^{\beta^{-\frac{1}{2}}})$ because $\beta < \beta$. For each U, define the *height-truncated leading coefficient*

$$F_{\Lambda,U}(y) := \sum_{\rho\in\mathcal{Z}_U m(\rho)\,W_{\Lambda,y}(\rho)=e^{y(\beta^{-\frac{1}{2}})\sum_{\rho\in\mathcal{Z}_U} m(\rho)\exp\left(\frac{(\rho-\frac{1}{2})^2}{\Lambda^2}\right)e^{iy\gamma}}}. \quad (5)$$

Then for each fixed U,

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = T^{\beta^{-\frac{1}{2}}\, F_{\Lambda,U}(y) + o\left(T^{\beta^{-\frac{1}{2}}}\right) + O_U(1),} \qquad (T\to\infty), \quad (6)$$

uniformly for y in any compact subinterval of a bounded interval.

Lemma 5.4 (Exponential-sum nondegeneracy). *If $F_{\Lambda,U}(y) = 0$ for all y in a nonempty open interval $I \subset \mathbb{R}$ and all $U > 0$, then $\mathcal{Z}^{=\emptyset}$.*

Proof. For each fixed U, write $a_\rho := m(\rho)\exp(((\rho - \frac{1}{2})^2)/\Lambda^2)$ and $\rho = \beta^{+i\gamma}$. Then

$$F_{\Lambda,U}(y) = e^{y(\beta^{-\frac{1}{2}})\sum_{\rho\in\mathcal{Z}_U} a_\rho\, e^{iy\gamma}.}$$

The prefactor never vanishes. An exponential polynomial $\sum a_\rho e^{iy\gamma}$ that vanishes on an interval must be identically zero; thus each coefficient $a_\rho = 0$ for all $\rho \in \mathcal{Z}_U$. Since this holds for all U and $m(\rho) \geq 1$, we have $\mathcal{Z}^{=\emptyset}$.

Converse direction of Theorem 1.1. Suppose $\mathcal{E}_{\sigma,\Lambda,y}(T) = o(T^{\frac{1}{2}-\sigma})$ for all $y \in I$. If $\beta^{>1/2}$, then for each fixed U, dividing (6) by $T^{\beta^{-\frac{1}{2}}}$ gives

$$\frac{\mathcal{E}_{\sigma,\Lambda,y}(T)}{T^{\beta^{-\frac{1}{2}}} = F_{\Lambda,U}(y) + o(1) + O_U(T^{-(\beta^{-\frac{1}{2}})).}}$$

Since $\beta^{-\frac{1}{2}>\frac{1}{2}-\sigma}$ (equivalently $1-\sigma-\beta^{<0}$), the assumed bound forces $F_{\Lambda,U}(y) \equiv 0$ on I for each U. By Lemma 5.4, $\mathcal{Z}^{=\emptyset}$, contradicting $\beta^{>1/2}$. Hence $\beta^{=\frac{1}{2}}$ and RH holds.

Corollary 5.5 (Open-interval echo–silence forces RH). *Assume that for every $\sigma \in (1/2, 1)$ there exists a nonempty open interval $I_\sigma \subset \mathbb{R}$ such that*

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = o\!\left(T^{\frac{1}{2}-\sigma}\right) \quad \textit{uniformly for } y \in I_\sigma.$$

Then RH holds.

Proof. If $\beta^* > 1/2$, choose σ with $1 - \sigma < \beta^* < \sigma$. By Theorem 5.1, $\mathcal{E}_{\sigma,\Lambda,y}(T) = T^{\beta^*-\frac{1}{2}} F_{\Lambda,U}(y) + o(T^{\beta^*-\frac{1}{2}})$ with a nontrivial exponential polynomial $F_{\Lambda,U}$, contradicting $o(T^{\frac{1}{2}-\sigma})$ on any open I_σ (since $\beta^* - \frac{1}{2} > 0$ while $\frac{1}{2} - \sigma < 0$). Hence $\beta^* = 1/2$.

6 Equivalent formulations and robustness

Corollary 6.1 (Uniform echo–silence). *RH holds iff for some (equivalently, every) $\sigma \in (\frac{1}{2}, 1)$ and $\Lambda > 0$ there is a nonempty open interval I with*

$$\sup_{y\in I} \limsup_{T\to\infty} T^{-(\frac{1}{2}-\sigma)} \left|\mathcal{E}_{\sigma,\Lambda,y}(T)\right| = 0.$$

Remark 6.2 (Choice of weight). Any entire, nonvanishing, even-in-$s-\frac{1}{2}$ factor with superpolynomial vertical decay may replace the Gaussian; the tilt $e^{y(s-1/2)}$ can be replaced by any nondegenerate one-parameter entire family $E_y(s)$ with $E_y(1-s) = E_{-y}(s)$ and $E_y(\rho)$ nontrivial in y. The proofs are unchanged.

7 Unconditional inputs and routes to vanishing

This paper isolates RH into the uniform vanishing of $\mathcal{E}_{\sigma,\Lambda,y}(T)$. Any unconditional method proving

$$\mathcal{E}_{\sigma,\Lambda,y}(T) = o\big(T^{\frac{1}{2}-\sigma}\big) \quad \text{as } T \to \infty,$$

for y in some interval yields RH by the converse direction of Theorem 1.1. Candidate routes (all independent of the specific choice of $W_{\Lambda,y}$):

- Mean-value bounds for ξ'/ξ with Gaussian weight (vertical large-sieve style).
- Dispersion/energy inequalities for coprime-filtered moments, converting off-diagonal control into echo-silence.
- Positivity methods: represent $\mathcal{E}$ as $\int |G_\sigma(s)|^2 \, d\mu_T$ with a positive measure $d\mu_T$ and show $o(T^{\frac{1}{2}-\sigma})$.

8 Final assembly

Unconditional results.

- The mirror functional $\mathcal{E}_{\sigma,\Lambda,y}(T)$ admits an absolutely convergent residue expansion (Theorem 1.1, Lemma 2.3)
- The expansion isolates the asymmetry from off-line zeros via the factor $(T^{\beta-\frac{1}{2}} - T^{\frac{1}{2}-\beta})$
- Exponential-sum nondegeneracy prevents accidental cancellation (Lemma 5.4)

Conditional results (assuming RH).

- If RH holds, then $\mathcal{E}_{\sigma,\Lambda,y}(T) \equiv 0$ for all $T \geq 2$ (Proposition 4.1)
- The converse requires only vanishing on an open interval in y (converse direction of Theorem 1.1)
- Thus RH $\Leftrightarrow$ uniform echo-silence across mirrors

Barriers and limits. This equivalence does not constitute a proof of RH. The required vanishing bound $\mathcal{E}_{\sigma,\Lambda,y}(T) = o(T^{1/2-\sigma})$ must be established by independent methods. The companion paper achieves this through coprime-diagonal analysis with Type I/II decomposition. Alternative routes include mean-value bounds, dispersion inequalities, or positivity methods (Section 7).

Appendix (optional): Critical line as channel

After unfolding to unit mean spacing, the pair-correlation kernel suggests the sine/sinc kernel

$$K(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Interpreting ordinates γ_k as samples and defining the field $\Phi(t) = \sum_k K(t,\gamma_k) e^{i\theta_k}$ gives a bandlimited reconstruction picture; the log-derivative ζ'/ζ spikes near zeros correspond to Fisher-information peaks. This heuristic plays no role in Sections 3–7.

References

[1] E. C. Titchmarsh (revised by D. R. Heath-Brown), *The Theory of the Riemann Zeta-Function*, 2nd ed., OUP.

[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, Vol. 53.

Paper Summary: Echo–Silence on the Critical Horizon and the Riemann Hypothesis

This paper introduces a novel approach to the Riemann Hypothesis (RH) by defining a **mirror functional** $\mathcal{E}_{\sigma,\Lambda,y}(T)$. This functional measures a quantifiable imbalance that would exist if any non-trivial zeta zero were located off the critical line $\Re s = \frac{1}{2}$. The argument is built using classical complex analysis and relies on the functional equation of the completed zeta, $\xi(s) = \xi(1-s)$.

Summation convention: all sums $\sum_\rho$ are taken as **symmetric height truncations** $|\Im\rho| \le U$ and limits $U \to \infty$ are justified by the convergence of the defining contour integrals (Gaussian vertical decay); boundary zeros are handled by indentation.

Key Components

- **Mirror Functional $\mathcal{E}$:** Defined as a difference of two contour integrals on mirror vertical lines, $\Re s = \sigma$ and $\Re s = 1-\sigma$. By residue calculus (with symmetric truncation) this equals a sum over non-trivial zeros $\rho = \beta + i\gamma$.
- **The Imbalance Term:** The core factor $(T^{\beta-\sigma} - T^{1-\beta-\sigma})$ is nonzero iff $\beta \neq \frac{1}{2}$.
- **Mirror Weight $W_{\Lambda,y}(s)$:** A Gaussian factor ensures uniform decay on vertical lines, and a **tilt** $e^{y(s-1/2)}$ (with $W_{\Lambda,y}(1-s) = W_{\Lambda,-y}(s)$) prevents accidental cancellation by separating ordinates via an exponential polynomial in y.

Main Theorem and Its Implications

The central theorem proves a direct equivalence:

1. **RH $\Rightarrow$ Echo-Silence:** If RH holds ($\beta = \frac{1}{2}$ for all zeros), each pair cancels in the truncated residue expansion, and the contour formulation yields $\mathcal{E}_{\sigma,\Lambda,y}(T) \equiv 0$ (hence also $O_{\sigma,\Lambda}(T^{-A})$ for every $A > 0$).
2. **Echo-Silence $\Rightarrow$ RH:** Conversely, if $\mathcal{E}_{\sigma,\Lambda,y}(T) = o(T^{1/2-\sigma})$ for **all y in some open interval**, then an exponential-polynomial argument forces the leading off-line coefficient to vanish, implying all zeros lie on $\Re s = \frac{1}{2}$.

This reframes RH as the unconditional vanishing of a single, precisely defined functional. Future work can target this vanishing via mean-value bounds, dispersion/positivity methods, or related analytic machinery.

The optional appendix offers a heuristic "channel" interpretation where zeros act as sampling points for a communication field. This viewpoint is **not used** in the proofs.